International Journal of Approximate Reasoning 122 (2020) 47–65

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Multi-granulation method for information fusion in multi-source decision information system

Lei Yang [a], Weihua Xu [b,*], Xiaoyan Zhang [b], Binbin Sang [c]

[a] School of Science, Chongqing University of Technology, Chongqing, 400054, PR China
[b] College of Artificial Intelligence, Southwest University, Chongqing, 400715, PR China
[c] Institute of Artificial Intelligence, School of Information Science and Technology, Southwest Jiaotong University, Chengdu, 611756, PR China

ARTICLE INFO

Article history:
Received 25 July 2019
Received in revised form 8 March 2020
Accepted 5 April 2020
Available online 13 May 2020

Keywords:
Multi-source decision information system
Decision support characteristic function
Decision related characteristic function
Fixed aggregation operator
Possible aggregation operator

ABSTRACT

Most of the existing multi-source fusion methods are to choose the most reliable information from the multi-source information system to form a single-source information system. Obviously, this process is accompanied by information loss. In order to solve this problem, the multi-granulation method of information fusion in multi-source decision information system is studied in this paper. Firstly, decision support characteristic function and decision related characteristic function are constructed. Secondly, a pair of aggregation operators, including fixed aggregation operator and possible aggregation operator, is defined through two characteristic functions. Meanwhile, the two cases when thresholds α and β take special values are discussed. Finally, the relevant properties of aggregation operators in different situations are proposed and proved. What is more, two groups of comparative experiments are carried out to illustrate the effect of the aggregation operators. The experimental results show that the proposed multi-source fusion method can always find a set of thresholds (α, β), which makes the fusion effect better than the mean fusion.

1. Introduction

Many complex information systems came into being with the advent of the era of big data. How to discover knowledge and extract rules from these complex information systems is an urgent problem to be solved. Rough set theory (RST) [1] provides us with a good idea. The RST was firstly proposed by researchers represented by polish mathematician Pawlak in 1982, and the first monograph [2] on RST was published in 1992. The RST [3,4] is an effective tool for dealing with incomplete information. At the same time, the RST is also a very effective method to deal with complex information systems. Traditional RST had been extended from different levels in order to solve different problems. The classical RST was established during the equivalence relation in the domain. However, the relation we study often fails to meet the standard of equivalence relation due to the lack of information and other reasons. For this reason, generalized rough sets based on general relations [5–11] were researched. Zhang and Yang proposed the three-way group decisions in interval-valued decision-theoretic rough sets by combining the aggregating inclusion measures [12].

As a kind of complex information system, multi-source decision information system (MsDIS) has been widely concerned by scholars. However, the rough set models mentioned above cannot solve the problem of knowledge discovery in MsDIS.

* Corresponding author.
E-mail address: chxuwh@gmail.com (W. Xu).

https://doi.org/10.1016/j.ijar.2020.04.003
0888-613X/

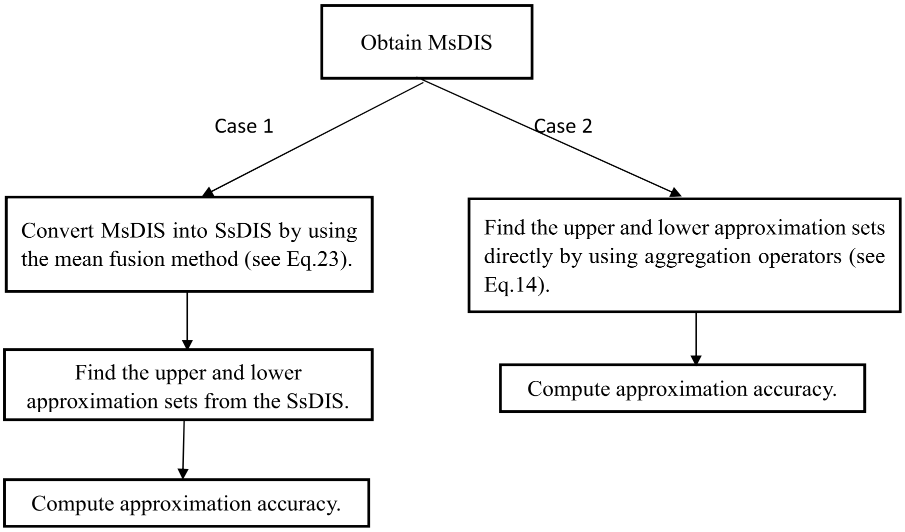

Fig. 1. Two cases of information fusion.

Because the equivalence relation in classical RST is considered as a granularity and a partition of universe can be regarded as a granularity structure from the perspective of granular computing [13,14]. Therefore, the classical RST is formed in a single granularity space under an equivalence relation. It can be seen that the above expansion of classical RST are carried out in single granularity space. However, there are multiple information tables in a MsDIS, and each information table corresponds to a granular structure. At this time, the traditional RST and its extended models are no longer applicable.

In order to apply RST more widely to complex information systems in real life, Qian et al. proposed multi-granulation rough sets [15,16]. In recent years, many scholars have also done a lot of researches on multi-granulation rough sets [17–23]. The multi-granulation rough set was studied by Yang et al. from the perspective of sensitive cost [24,25]. Hu et al. studied the dynamic approximation updating of multi-granulation rough sets by using matrix method, and realized the decision-making under dynamic conditions [26]. Li et al. analyzed the relation between multi-granulation rough sets and concept lattices from the perspective of rule acquisition [27]. Qian et al. combined with the idea of probability theory to propose a multi-granulation decision-theoretic rough set model [28]. Lin et al. replaced the classical equivalence relation with the neighborhood relationship to expand the multi-granulation rough set theory [29]. Zhang et al. studied the local multi-granulation DTRS in the order information system [30]. Lin et al. proposed a multi-threshold tolerance relation-based multi-granulation rough sets for incomplete interval-valued decision information systems [31]. Yu et al. proposed a double-quantitative decision making theory in a multi-granulation approximate space [32]. Xu et al. studied multigranulation fuzzy rough set [33,34]. Zhan and Xu proposed a method of solving the multi-criterion group decision [35]. Qian et al. proposed a multiple different thresholds based generalized multigranulation sequential three-way decision model [36]. Sang et al. proposed Generalized Multi-granulation double-quantitative DTRS in multi-source information system [37].

In recent years, information fusion [38–42], as an interesting method to deal with multi-source information system (MsIS), has been widely noticed. Most of the existing fusion methods transform MsIS into single-source information system (SsIS) and then carry out knowledge discovery and rule extraction. Xu et al. considered information from the perspective of triangular fuzzy granular, and realized the selection of information sources through the internal and external confidence [43]. Huang et al. studied the dynamic fusion of interval value data sets starting from trapezoidal fuzzy granular [44]. Guo et al. proposed the reduction method of multi-source information systems: consistent reduction and reduction after fusion [45]. Because of the ambiguity and incompleteness of the source, Xu et al. used conditional entropy to select information sources, and then merged the multi-source information table into a table [46,47]. The principle of those information fusion method is similar to that of the mean fusion represented by case 1 in Fig. 1. The common feature of these fusion methods is that before knowledge discovery or rule extraction, the most reliable information in the MsIS is selected and integrated into a SsIS through different methods. It is worth noting that there is information loss in multi-source fusion by these methods. Therefore, how to discover knowledge from MsDIS without information loss is the motivation of this paper.

In 2018, Wei and Liang discussed the overview of information fusion from the perspective of RST [48]. It is known from literature [48] that multi-granulation rough set is an effective tool for information fusion. Lin et al. studied information fusion by combining multi-granulation and evidence theory [49]. Che et al. studied the information fusion and related digital characteristics of multi-source information system from the perspective of multi-granulation [50]. Similarly, this paper also studies the information fusion method of MsDIS from the perspective of multi-granulation rough set. We consider the equivalence relation formed by all conditional attributes in each information source as a granularity, and then q information sources form q granularity. Then the aggregation operators are constructed through multi-granulation rough set. The main idea of this paper is illustrated in a more concise manner, which is shown in case 2 in Fig. 1. The research results show that we can also classify objects without merging multi-source decision information tables into one table. In addition, one can see from the research results that the higher the correlation between sources, the better the effect of our proposed fusion method.

The main contributions of this paper are as follows. First, we define the decision support characteristic function and decision related characteristic function to describe the inclusion relation and the intersection non-empty relation between equivalent class and concept in each information source. Second, three groups of aggregation operators are built through two characteristic functions. Meanwhile, the properties of aggregation operators are proposed and proved. Third, we propose two comparison algorithms, and experiment with 9 data sets on UCI shows that the fusion method proposed in this paper can realize direct knowledge discovery without information loss from MsDIS.

The rest of this paper is divided into the following sections. In section 2, we mainly review related works like the approximate accuracy of concepts, multi-granulation rough sets, and multi-source decision information systems. In section 3, firstly, decision support characteristic function and decision related characteristic function are constructed; secondly, three sets of aggregation operators are defined through these two functions; finally, the properties of the aggregation operators are discussed. In order to prove the validity of our proposed aggregation operators, we download 9 data sets in the machine learning database and conduct a series of experiments in section 4. In section 5, the conclusion of this paper was given.

2. Preliminaries

As we study the multi-granulation approach of information fusion in multi-source decision information systems, we briefly review the basic concepts related to rough sets, support characteristic function, multi-granulation rough sets and multi-source decision information systems in this section.

2.1. Pawlak's rough sets

In general, we use a quad $I = (U, AT, V, F)$ to represent an information system, where U is a non-empty finite set of objects, called the universe; AT is a non-empty finite set of attributes; $V = \bigcup_{a\in AT} V_a$, V_a is the domain of the attribute a; $F = \{f \mid U \times AT \to V\}$ is the set of information functions, $f(x, a) \in V_a$ $(x \in U,\ a \in AT)$. Let X, Y be two non-empty classical sets, $X \times Y = \{(x, y) : x \in X, y \in Y\}$ is Cartesian product of X and Y. For any $R \subseteq X \times Y$, R is called the binary relation from X to Y, abbreviated as a relation.

Definition 2.1. [51] Let $R \subseteq X \times X$, R is an equivalence relation on X if R satisfy:

(1) reflexive, i.e. $\forall x \in X$, $(x, x) \in R$;
(2) symmetric, i.e. $\forall x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$;
(3) transitive, i.e. $\forall x, y, z \in X$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

If R is an equivalence relation on X, for any $x \in X$, the equivalent class represented by x is denoted as $[x]_R = \{y \in X : (x, y) \in R\}$, abbreviated as an equivalent class.

Definition 2.2. [51] Let R be an equivalent relation on the domain U, and (U, R) is an approximate space. For a target set X $(X \subseteq U)$, the lower and upper approximation sets of the target set X for R are defined as follows:

$$\underline{R}(X) = \cup\{x \in U : [x]_R \subseteq X\}, \quad \overline{R}(X) = \cup\{x \in U : [x]_R \cap X \neq \emptyset\}. \tag{1}$$

If $\underline{R}(X) = \overline{R}(X)$, then X is definable or accurate for R; otherwise, X is rough about R.

Definition 2.3. [51] Let (U, R) be an approximate space, $X \subseteq U$ $(X \neq \emptyset)$. The approximate accuracy and roughness of the target set X are defined as follows:

$$\varrho_R(X) = \frac{|\underline{R}(X)|}{|\overline{R}(X)|}, \quad \rho_R(X) = 1 - \varrho_R(X). \tag{2}$$

They are used to measure the uncertainty of X through the numerical characteristics of X. $|A|$ represents the number of elements in the set A.

For a decision system $I = (U, AT, V, F)$, where $AT = C \cup D$, C is a conditional attribute set, and D is a decision attribute set. $U/D = \{D_1, D_2, \cdots, D_m\}$ is the division of the domain U on the decision attribute set, and A is a subset of the conditional attribute set. The lower and upper approximation sets of the partition U/D are defined as follows:

$$\underline{A}(U/D) = \underline{A}(D_1) \cup \underline{A}(D_2) \cup \cdots \cup \underline{A}(D_m), \quad \overline{A}(U/D) = \overline{A}(D_1) \cup \overline{A}(D_2) \cup \cdots \cup \overline{A}(D_m). \tag{3}$$

The definition of approximate accuracy and roughness for U/D is shown as follows:

$$\varrho_A(U/D) = \frac{\sum_{D_j \in U/D} |\underline{A}(D_j)|}{\sum_{D_j \in U/D} |\overline{A}(D_j)|}, \quad \rho_A(U/D) = 1 - \varrho_A(U/D). \tag{4}$$

2.2. Support characteristic function and generalized multi-granulation rough set model

The support characteristic function is defined by the relation between the equivalence class and the concept set, then the generalized multi-granulation rough set model is defined by the support characteristic function.

Definition 2.4. [52] Let $I = (U, AT, V, F)$ be an information system, $A_i \subseteq AT$, $i = 1, 2, \cdots, s$ $(s \leq 2^{|AT|})$. For any X $(X \subseteq U)$, the support characteristic function of x for X is defined as follows:

$$\mathcal{S}_X^{A_i}(x) = \begin{cases} 1, & [x]_{A_i} \subseteq X, \\ 0, & otherwise, \end{cases} \quad (i \leq 2^{|AT|}). \tag{5}$$

The support characteristic function $\mathcal{S}_X^{A_i}(x)$ is used to describe the inclusion relation between equivalence class $[x]_{A_i}$ and concept X, which indicates whether object x accurately supports X by A_i.

Definition 2.5. [52] Let $I = (U, AT, V, F)$ be an information system, $A_i \subseteq AT$, $i = 1, 2, \cdots, s$ $(s \leq 2^{|AT|})$, $\beta \in (0.5, 1]$, $\mathcal{S}_X^{A_i}(x)$ is the support characteristic function of x for X. For any $X \subseteq U$, the lower approximation and upper approximation of X for $\sum_{i=1}^{s} A_i$ are defined as follows:

$$\underline{GM}_{\sum_{i=1}^{s} A_i}(X)_\beta = \left\{ x \in U \left| \frac{\sum_{i=1}^{s} \mathcal{S}_X^{A_i}(x)}{s} \geq \beta \right. \right\}, \quad \overline{GM}_{\sum_{i=1}^{s} A_i}(X)_\beta = \left\{ x \in U \left| \frac{\sum_{i=1}^{s} (1 - \mathcal{S}_{X^c}^{A_i}(x))}{s} > 1 - \beta \right. \right\}. \tag{6}$$

The target set X is definable iff $\underline{GM}_{\sum_{i=1}^{s} A_i}(X)_\beta = \overline{GM}_{\sum_{i=1}^{s} A_i}(X)_\beta$, otherwise, X is a rough set. We denote this generalized multi-granulation rough set model as GMGRS, and β is the information level for $\sum_{i=1}^{s} A_i$.

2.3. Pessimistic and optimistic multi-granulation rough set models

When the domain is divided by multiple equivalence relations, combined with the idea of single-granulation rough set, the multi-granulation rough set is divided into pessimistic multi-granulation rough set and optimistic multi-granulation rough set.

Definition 2.6. [52] Let $I = (U, AT, V, F)$ be an information system, $A_i \subseteq AT$, $i = 1, 2, \cdots, s$ $(s \leq 2^{|AT|})$. For any $X \subseteq U$, the pessimistic lower approximation and pessimistic upper approximation of X for $\sum_{i=1}^{s} A_i$ are defined as follows:

$$\underline{PM}_{\sum_{i=1}^{s} A_i}(X) = \left\{ x \in U \left| \wedge_{i=1}^{s} ([x]_{A_i} \subseteq X) \right. \right\}, \quad \overline{PM}_{\sum_{i=1}^{s} A_i}(X) = \left\{ x \in U \left| \vee_{i=1}^{s} ([x]_{A_i} \cap X \neq \emptyset) \right. \right\}, \tag{7}$$

where "$\vee$"and "$\wedge$"denote "or"and "and", respectively. If you consider pessimistic multi-granulation from the perspective of support characteristic function, the formula (7) has another form as follows:

$$\underline{PM}_{\sum_{i=1}^{s} A_i}(X) = \left\{ x \in U \left| \frac{\sum_{i=1}^{s} \mathcal{S}_X^{A_i}(x)}{s} \geq 1 \right. \right\}, \quad \overline{PM}_{\sum_{i=1}^{s} A_i}(X) = \left\{ x \in U \left| \frac{\sum_{i=1}^{s} (1 - \mathcal{S}_{X^c}^{A_i}(x))}{s} > 0 \right. \right\}. \tag{8}$$

Definition 2.7. [52] Let $I = (U, AT, V, F)$ be an information system, $A_i \subseteq AT$, $i = 1, 2, \cdots, s$ $(s \leq 2^{|AT|})$. For any $X \subseteq U$, the optimistic lower approximation and optimistic upper approximation of X for $\sum_{i=1}^{s} A_i$ are defined as follows:

$$\underline{OM}_{\sum_{i=1}^{s} A_i}(X) = \left\{ x \in U \left| \vee_{i=1}^{s} ([x]_{A_i} \subseteq X) \right. \right\}, \quad \overline{OM}_{\sum_{i=1}^{s} A_i}(X) = \left\{ x \in U \left| \wedge_{i=1}^{s} ([x]_{A_i} \cap X \neq \emptyset) \right. \right\}, \tag{9}$$

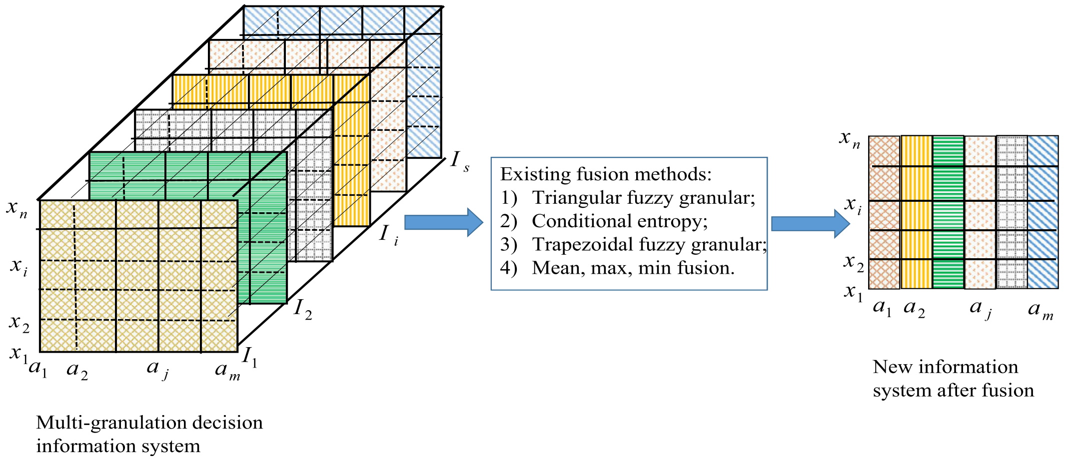

Fig. 2. The fusion process of existing fusion methods.

where "$\vee$"and "$\wedge$"denote "or"and "and", respectively. If you consider optimistic multi-granulation from the perspective of support characteristic function, the formula (9) has another form as follows:

$$\underline{OM}_{\sum\limits_{i=1}^{s} A_i}(X) = \left\{ x \in U \left| \frac{\sum\limits_{i=1}^{s} \mathcal{S}_X^{A_i}(x)}{s} > 0 \right. \right\}, \quad \overline{OM}_{\sum\limits_{i=1}^{s} A_i}(X) = \left\{ x \in U \left| \frac{\sum\limits_{i=1}^{s} (1 - \mathcal{S}_{X^c}^{A_i}(x))}{s} \geq 1 \right. \right\}. \tag{10}$$

2.4. Multi-source decision information systems

The same structure of multi-source information system was obtained by acquiring information about the same object and the same attribute from different information sources. The definition of the multi-source information system is given as follows.

Definition 2.8. [49] A multi-source information system is composed by a plurality of quads in the form of $IS_i = (U, AT, V_i, F_i)$, where U represents the entirety of the object, AT represents the attribute set, V_i represents the range of attribute values under source i, and F_i represents the corresponding relation between the object and the feature under the source i. In general, A multi-source information system (MsIS) is expressed as follows:

$$M_s IS = \{IS_1, IS_2, \ldots, IS_q\}. \tag{11}$$

Similarly, a multi-source decision information system (MsDIS) is composed by multiple single information systems in the form of $DIS_i = (U, AT, V_i, F_i, DT, D_i, G_i)$, where the meanings of U, AT, V_i and F_i are consistent with those described in the MsIS, DT represents the decision attribute set, D_i represents the range of the decision attribute value under source i, and G_i represents the corresponding relation between the object and the decision attribute under source i. In general, the MsDIS is expressed as follows:

$$M_S DIS = \{DIS_1, DIS_2, \ldots, DIS_q\}. \tag{12}$$

3. Multi-granulation method for information fusion in multi-source decision information system

The traditional information fusion method regards the MsDIS as an information box, and then fuses the information box into an information table by means of fuzzy granules and conditional entropy. The traditional fusion process is shown in Fig. 2. Finally, the fused information table is used for decision-making and rule extraction. In order to discover knowledge directly from multi-source decision data sets, the fusion of multi-source information was constructed from the view of multi-granulation in this section.

3.1. Fixed aggregation operator and possible aggregation operator of MsDIS

In this subsection, inspired by the idea of RST, we first define the decision support characteristic function and the decision related characteristic function, and then construct the information aggregation operators (the fixed aggregation operator and possible aggregation operator) of the MsDIS from the perspective of generalized multi-granulation, and finally we propose and prove the related properties of aggregation operators.

Definition 3.1. Given a multi-source decision information system $M_S DIS = \{DIS_1, DIS_2, \ldots, DIS_q\}$, where $DIS_i = (U, AT, V_i, F_i, DT, D_i, G_i)$ $(i = 1, 2, \ldots, q)$. $U/D = \{D_1, D_2, \ldots, D_m\}$, for each $D_j \in U/D$ $(j = 1, 2, \ldots, m)$, the decision support characteristic function and decision related characteristic function of x for D_j are defined as follows:

$$\mathcal{DS}_{D_j}^{DIS_i}(x) = \begin{cases} 1, [x]_{DIS_i} \subseteq D_j, \\ 0, otherwise. \end{cases} \quad (i \le q), \quad \mathcal{DR}_{D_j}^{DIS_i}(x) = \begin{cases} 1, [x]_{DIS_i} \cap D_j \neq \emptyset, \\ 0, otherwise. \end{cases} \quad (i \le q). \tag{13}$$

The decision support characteristic function $\mathcal{DS}_{D_j}^{DIS_i}(x)$ is used to describe the inclusion relation between equivalence class $[x]_{DIS_i}$ and concept D_j, which indicates whether object x accurately supports D_j by DIS_i. The decision related characteristic function $\mathcal{DR}_{D_j}^{DIS_i}(x)$ is used to describe the intersection non-empty relation between equivalence class $[x]_{DIS_i}$ and concept D_j, which indicates whether object x possibly supports D_j by DIS_i.

Definition 3.2. Let $M_S DIS = \{DIS_1, DIS_2, \ldots, DIS_q\}$ be a MsDIS, where $DIS_i = (U, AT, V_i, F_i, DT, D_i, G_i)$ $(i = 1, 2, \ldots, q)$. α, β are two thresholds which satisfied $\alpha + \beta = 1$ and $\alpha \in (0, 1]$, $\beta \in [0, 1)$. $U/D = \{D_1, D_2, \ldots, D_m\}$, $\mathcal{DS}_{D_j}^{DIS_i}(x)$ is the decision support characteristic function of x for D_j, $\mathcal{DR}_{D_j}^{DIS_i}(x)$ is the decision related characteristic function of x for D_j. For any $D_j \in U/D$ $(j = 1, 2, \ldots, m)$, the fixed aggregation operator and possible aggregation operator of MsDIS are defined as follows:

$$\bigotimes_{i=1}^{q} DIS_i(D_j)_\alpha = \left\{ x \in U \,\middle|\, \frac{\sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} \ge \alpha \right\}, \quad \bigoplus_{i=1}^{q} DIS_i(D_j)_\beta = \left\{ x \in U \,\middle|\, \frac{\sum_{i=1}^{q} \mathcal{DR}_{D_j}^{DIS_i}(x)}{q} > \beta \right\}. \tag{14}$$

Definition 3.3. The definition of fusion approximate accuracy and fusion roughness when dividing U/D at thresholds (α, β) is shown as follows:

$$\varrho_{MSD}(U/D)_{(\alpha,\beta)} = \frac{\sum_{D_j \in U/D} \left| \bigotimes_{i=1}^{q} DIS_i(D_j)_\alpha \right|}{\sum_{D_j \in U/D} \left| \bigoplus_{i=1}^{q} DIS_i(D_j)_\beta \right|}, \quad \rho_{MSD}(U/D)_{(\alpha,\beta)} = 1 - \varrho_{MSD}(U/D)_{(\alpha,\beta)}. \tag{15}$$

Property 3.1. *Let $M_S DIS = \{DIS_1, DIS_2, \ldots, DIS_q\}$ be a MsDIS, where $DIS_i = (U, AT, V_i, F_i, DT, D_i, G_i)$ $(i = 1, 2, \ldots, q)$. α, β are two thresholds which satisfied $\alpha + \beta = 1$ and $\alpha \in (0, 1]$, $\beta \in [0, 1)$. $U/D = \{D_1, D_2, \ldots, D_m\}$, for any $D_j \in U/D$ $(j = 1, 2, \ldots, m)$, the following properties of aggregation operators are true.*

(1) $\bigotimes_{i=1}^{q} DIS_i(D_j)_\alpha \subseteq D_j \subseteq \bigoplus_{i=1}^{q} DIS_i(D_j)_\beta$;

(2) $\bigotimes_{i=1}^{q} DIS_i(\emptyset)_\alpha = \bigoplus_{i=1}^{q} DIS_i(\emptyset)_\beta = \emptyset$;

(3) $\bigotimes_{i=1}^{q} DIS_i(U)_\alpha = \bigoplus_{i=1}^{q} DIS_i(U)_\beta = U$;

(4) $\forall D_i, D_j \in U/D$, $\bigotimes_{i=1}^{q} DIS_i(D_i \cap D_j)_\alpha \subseteq \bigotimes_{i=1}^{q} DIS_i(D_i)_\alpha \cap \bigotimes_{i=1}^{q} DIS_i(D_j)_\alpha$,
$\bigoplus_{i=1}^{q} DIS_i(D_i \cap D_j)_\beta \subseteq \bigoplus_{i=1}^{q} DIS_i(D_i)_\beta \cap \bigoplus_{i=1}^{q} DIS_i(D_j)_\beta$;

(5) $\forall D_i, D_j \in U/D$, $\bigotimes_{i=1}^{q} DIS_i(D_i \cup D_j)_\alpha \supseteq \bigotimes_{i=1}^{q} DIS_i(D_i)_\alpha \cup \bigotimes_{i=1}^{q} DIS_i(D_j)_\alpha$,
$\bigoplus_{i=1}^{q} DIS_i(D_i \cup D_j)_\beta \supseteq \bigoplus_{i=1}^{q} DIS_i(D_i)_\beta \cup \bigoplus_{i=1}^{q} DIS_i(D_j)_\beta$.

Proof. (1) For any $x \in \bigotimes_{i=1}^{q} DIS_i(D_j)_\alpha$, we can know $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} \geq \alpha$. Because of $\alpha \in (0, 1]$, so $\exists\ i \leq q$, s.t $[x]_{DIS_i} \subseteq D_j$. Known by the arbitrariness of x, $\bigotimes_{i=1}^{q} DIS_i(D_j)_\alpha \subseteq D_j$. For any $x \in D_j$, we can know $[x]_{DIS_i} \cap D_j \neq \emptyset$. Thus $\frac{\sum_{i=1}^{q} \mathcal{DR}_{D_j}^{DIS_i}(x)}{q} = 1 > \beta$, $x \in \bigoplus_{i=1}^{q} DIS_i(D_j)_\beta$. Known by the arbitrariness of x, $D_j \subseteq \bigoplus_{i=1}^{q} DIS_i(D_j)_\beta$. Therefore, the property (1) has been proved.

(2) Through the definition of decision support characteristic function and decision related characteristic function, we can gain $\mathcal{DS}_{\emptyset}^{DIS_i}(x) = 0, \mathcal{DR}_{\emptyset}^{DIS_i}(x) = 0$. So we can obtain the following two equations:

$$\bigotimes_{i=1}^{q} DIS_i(\emptyset)_\alpha = \left\{ x \in U \left| \frac{\sum_{i=1}^{q} \mathcal{DS}_{\emptyset}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} 0}{q} = 0 \geq \alpha \right. \right\} = \emptyset,$$

$$\bigoplus_{i=1}^{q} DIS_i(\emptyset)_\beta = \left\{ x \in U \left| \frac{\sum_{i=1}^{q} \mathcal{DR}_{\emptyset}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} 0}{q} = 0 > \beta \right. \right\} = \emptyset.$$

(3) Through the definition of decision support characteristic function and decision related characteristic function, we can obtain $\mathcal{DS}_{U}^{DIS_i}(x) = 1, \mathcal{DR}_{U}^{DIS_i}(x) = 1$. So we can gain the following two equations:

$$\bigotimes_{i=1}^{q} DIS_i(U)_\alpha = \left\{ x \in U \left| \frac{\sum_{i=1}^{q} \mathcal{DS}_{U}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} 1}{q} = 1 \geq \alpha \right. \right\} = U,$$

$$\bigoplus_{i=1}^{q} DIS_i(U)_\beta = \left\{ x \in U \left| \frac{\sum_{i=1}^{q} \mathcal{DR}_{U}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} 1}{q} = 1 > \beta \right. \right\} = U.$$

(4) $\forall x \in \bigotimes_{i=1}^{q} DIS_i(D_i \cap D_j)_\alpha$, we have

$$\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i \cap D_j}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x) \wedge \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} \geq \alpha.$$

So $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x) \wedge \sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} \geq \frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x) \wedge \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} \geq \alpha$. Thus $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x)}{q} \geq \alpha$ and $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} \geq \alpha$, i.e. $x \in \bigotimes_{i=1}^{q} DIS_i(D_i)_\alpha$ and $x \in \bigotimes_{i=1}^{q} DIS_i(D_j)_\alpha$. Therefore $x \in \bigotimes_{i=1}^{q} DIS_i(D_i)_\alpha \cap \bigotimes_{i=1}^{q} DIS_i(D_j)_\alpha$. Known by the arbitrariness of x, $\forall D_i, D_j \in U/D$, $\bigotimes_{i=1}^{q} DIS_i(D_i \cap D_j)_\alpha \subseteq \bigotimes_{i=1}^{q} DIS_i(D_i)_\alpha \cap \bigotimes_{i=1}^{q} DIS_i(D_j)_\alpha$. The same we can prove $\bigoplus_{i=1}^{q} DIS_i(D_i \cap D_j)_\beta \subseteq \bigoplus_{i=1}^{q} DIS_i(D_i)_\beta \cap \bigoplus_{i=1}^{q} DIS_i(D_j)_\beta$.

(5) $\forall x \in \bigotimes_{i=1}^{q} DIS_i(D_i)_\alpha \cup \bigotimes_{i=1}^{q} DIS_i(D_j)_\alpha$, i.e. $x \in \bigotimes_{i=1}^{q} DIS_i(D_i)_\alpha$ or $x \in \bigotimes_{i=1}^{q} DIS_i(D_j)_\alpha$, by definition we have $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x)}{q} \geq \alpha$ or $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} \geq \alpha$. What's more, $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i \cup D_j}^{DIS_i}(x)}{q} \geq \left(\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x)}{q} \vee \frac{\sum_{i=1}^{q} \mathcal{DR}_{D_j}^{DIS_i}(x)}{q} \right) \geq \alpha$, so $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i \cup D_j}^{DIS_i}(x)}{q} \geq \alpha$, i.e. $x \in$

$\bigotimes_{i=1}^{q} DIS_i(D_i \cup D_j)_\alpha$. Known by the arbitrariness of x, $\forall D_i, D_j \in U/D$, $\bigotimes_{i=1}^{q} DIS_i(D_i \cup D_j)_\alpha \supseteq \bigotimes_{i=1}^{q} DIS_i(D_i)_\alpha \cup \bigotimes_{i=1}^{q} DIS_i(D_j)_\alpha$. Similarly, we can prove $\bigoplus_{i=1}^{q} DIS_i(D_i \cup D_j)_\beta \supseteq \bigoplus_{i=1}^{q} DIS_i(D_i)_\beta \cup \bigoplus_{i=1}^{q} DIS_i(D_j)_\beta$.

Property 3.2. *Let $M_S DIS = \{DIS_1, DIS_2, \ldots, DIS_q\}$ be a MsDIS, where $DIS_i = (U, AT, V_i, F_i, DT, D_i, G_i)$ $(i = 1, 2, \ldots, q)$. α, β are two thresholds which satisfied $\alpha + \beta = 1$ and $\alpha \in (0, 1]$, $\beta \in [0, 1)$, $U/D = \{D_1, D_2, \ldots, D_m\}$. For different levels of information ($\alpha_1 \le \alpha_2$), the following properties are established.*

(1) $\bigotimes_{i=1}^{q} DIS_i(D_j)_{\alpha_2} \subseteq \bigotimes_{i=1}^{q} DIS_i(D_j)_{\alpha_1}$;
(2) $\bigoplus_{i=1}^{q} DIS_i(D_j)_{\beta_1} \subseteq \bigoplus_{i=1}^{q} DIS_i(D_j)_{\beta_2}$.

Proof. (1) For any $x \in \bigotimes_{i=1}^{q} DIS_i(D_j)_{\alpha_2}$, we can know $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} \ge \alpha_2$. Because of $\alpha_1 \le \alpha_2$, then $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} \ge \alpha_1$, thus $x \in \bigotimes_{i=1}^{q} DIS_i(D_j)_{\alpha_1}$. Known by the arbitrariness of x, $\bigotimes_{i=1}^{q} DIS_i(D_j)_{\alpha_2} \subseteq \bigotimes_{i=1}^{q} DIS_i(D_j)_{\alpha_1}$.

(2) For any $x \in \bigoplus_{i=1}^{q} DIS_i(D_j)_{\beta_1}$, we can know $\frac{\sum_{i=1}^{q} \mathcal{DR}_{D_j}^{DIS_i}(x)}{q} > \beta_1$. Because of $\alpha + \beta = 1$, so $\beta_1 = 1 - \alpha_1, \beta_2 = 1 - \alpha_2$, and $\alpha_1 \le \alpha_2$, then $\beta_1 \ge \beta_2$, therefore $\frac{\sum_{i=1}^{q} \mathcal{DR}_{D_j}^{DIS_i}(x)}{q} > \beta_2$, i.e. $x \in \bigoplus_{i=1}^{q} DIS_i(D_j)_{\beta_2}$. Known by the arbitrariness of x, $\bigoplus_{i=1}^{q} DIS_i(D_j)_{\beta_1} \subseteq \bigoplus_{i=1}^{q} DIS_i(D_j)_{\beta_2}$.

3.2. Pessimistic fixed aggregation operator and pessimistic possible aggregation operator of MsDIS

In this subsection, the pessimistic fixed aggregation operator and pessimistic possible aggregation operator of MsDIS are constructed from the angle of pessimistic multi-granulation rough set.

Definition 3.4. Let $M_S DIS = \{DIS_1, DIS_2, \ldots, DIS_q\}$ be a MsDIS, where $DIS_i = (U, AT, V_i, F_i, DT, D_i, G_i)$ $(i = 1, 2, \ldots, q)$. $U/D = \{D_1, D_2, \ldots, D_m\}$, for any $D_j \in U/D$ $(j = 1, 2, \ldots, m)$, the pessimistic fixed aggregation operator and pessimistic possible aggregation operator of MsDIS are defined as follows:

$$\bigotimes_{i=1}^{q}{}^{P} DIS_i(D_j) = \left\{x \in U \,\middle|\, \wedge_{i=1}^{q} ([x]_{DIS_i} \subseteq D_j)\right\} = \left\{x \in U \,\middle|\, \frac{\sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} = 1\right\}, \tag{16}$$

$$\bigoplus_{i=1}^{q}{}^{P} DIS_i(D_j) = \left\{x \in U \,\middle|\, \vee_{i=1}^{q} ([x]_{DIS_i} \cap D_j \neq \emptyset)\right\} = \left\{x \in U \,\middle|\, \frac{\sum_{i=1}^{q} \mathcal{DR}_{D_j}^{DIS_i}(x)}{q} > 0\right\}. \tag{17}$$

Definition 3.5. The definition of pessimistic fusion approximate accuracy and pessimistic fusion roughness for dividing U/D is shown as follows:

$$\varrho_{MSD}(U/D)^P = \frac{\sum_{D_j \in U/D} \left|\bigotimes_{i=1}^{q}{}^{P} DIS_i(D_j)\right|}{\sum_{D_j \in U/D} \left|\bigoplus_{i=1}^{q}{}^{P} DIS_i(D_j)\right|}, \quad \rho_{MSD}(U/D)^P = 1 - \varrho_{MSD}(U/D)^P. \tag{18}$$

Property 3.3. *Let $M_S DIS = \{DIS_1, DIS_2, \ldots, DIS_q\}$ be a MsDIS, where $DIS_i = (U, AT, V_i, F_i, DT, D_i, G_i)$ $(i = 1, 2, \ldots, q)$. $U/D = \{D_1, D_2, \ldots, D_m\}$, for any $D_j \in U/D$ $(j = 1, 2, \ldots, m)$, the following properties are true.*

(1) $\bigotimes_{i=1}^{q} {}^{P} DIS_i(D_j) \subseteq D_j \subseteq \bigoplus_{i=1}^{q} {}^{P} DIS_i(D_j)$;

(2) $\bigotimes_{i=1}^{q} {}^{P} DIS_i(\emptyset) = \bigoplus_{i=1}^{q} {}^{P} DIS_i(\emptyset) = \emptyset$;

(3) $\bigotimes_{i=1}^{q} {}^{P} DIS_i(U) = \bigoplus_{i=1}^{q} {}^{P} DIS_i(U) = U$;

(4) $\forall D_i, D_j \in U/D, \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_i \cap D_j) = \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_i) \cap \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_j)$,
$\bigoplus_{i=1}^{q} {}^{P} DIS_i(D_i \cap D_j) \subseteq \bigoplus_{i=1}^{q} {}^{P} DIS_i(D_i) \cap \bigoplus_{i=1}^{q} {}^{P} DIS_i(D_j)$;

(5) $\forall D_i, D_j \in U/D, \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_i \cup D_j) \supseteq \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_i) \cup \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_j)$,
$\bigoplus_{i=1}^{q} {}^{P} DIS_i(D_i \cup D_j) = \bigoplus_{i=1}^{q} {}^{P} DIS_i(D_i) \cup \bigoplus_{i=1}^{q} {}^{P} DIS_i(D_j)$.

Proof. (1) For any $x \in \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_j)$, we can know $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} = 1$. i.e. $\forall i \leq q$, s.t $[x]_{DIS_i} \subseteq D_j$. Known by the arbitrariness of x, $\bigotimes_{i=1}^{q} {}^{P} DIS_i(D_j) \subseteq D_j$. Similarly, we can prove $D_j \subseteq \bigoplus_{i=1}^{q} {}^{P} DIS_i(D_j)$. Therefore, $\bigotimes_{i=1}^{q} {}^{P} DIS_i(D_j) \subseteq D_j \subseteq \bigoplus_{i=1}^{q} {}^{P} DIS_i(D_j)$.

(2) Through the definition of decision support characteristic function and decision related characteristic function, we can obtain $\mathcal{DS}_{\emptyset}^{DIS_i}(x) = 0, \mathcal{DR}_{\emptyset}^{DIS_i}(x) = 0$. So we can gain the following two equations:

$$\bigotimes_{i=1}^{q} {}^{P} DIS_i(\emptyset) = \left\{ x \in U \left| \frac{\sum_{i=1}^{q} \mathcal{DS}_{\emptyset}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} 0}{q} = 0 \geq 1 \right. \right\} = \emptyset,$$

$$\bigoplus_{i=1}^{q} {}^{P} DIS_i(\emptyset) = \left\{ x \in U \left| \frac{\sum_{i=1}^{q} \mathcal{DR}_{\emptyset}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} 0}{q} = 0 > 0 \right. \right\} = \emptyset.$$

(3) Through the definition of decision support characteristic function and decision related characteristic function, we can obtain $\mathcal{DS}_{U}^{DIS_i}(x) = 1, \mathcal{DR}_{U}^{DIS_i}(x) = 1$. So we can obtain the following two equations:

$$\bigotimes_{i=1}^{q} {}^{P} DIS_i(U) = \left\{ x \in U \left| \frac{\sum_{i=1}^{q} \mathcal{DS}_{U}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} 1}{q} = 1 \geq 1 \right. \right\} = U,$$

$$\bigoplus_{i=1}^{q} {}^{P} DIS_i(U) = \left\{ x \in U \left| \frac{\sum_{i=1}^{q} \mathcal{DR}_{U}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} 1}{q} = 1 > 0 \right. \right\} = U.$$

(4) $\forall x \in \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_i \cap D_j) \Leftrightarrow \frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i \cap D_j}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x) \wedge \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} = 1 \Leftrightarrow \frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x)}{q} = 1$ and $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} = 1 \Leftrightarrow$ $x \in \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_i)$ and $x \in \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_j) \Leftrightarrow x \in \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_i) \cap \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_j)$. Known by the arbitrariness of x, $\forall D_i, D_j \in U/D, \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_i \cap D_j) = \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_i) \cap \bigotimes_{i=1}^{q} {}^{P} DIS_i(D_j)$.

For any $x \in \bigoplus_{i=1}^{q}{}^{P} DIS_i(D_i \cap D_j)$, we have $\frac{\sum_{i=1}^{q} \mathcal{DR}_{D_i \cap D_j}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} \mathcal{DR}_{D_i}^{DIS_i}(x) \wedge \mathcal{DR}_{D_j}^{DIS_i}(x)}{q} > 0$. So $\frac{\sum_{i=1}^{q} \mathcal{DR}_{D_i}^{DIS_i}(x)}{q} > 0$ and $\frac{\sum_{i=1}^{q} \mathcal{DR}_{D_j}^{DIS_i}(x)}{q} > 0$. $x \in \bigoplus_{i=1}^{q}{}^{P} DIS_i(D_i)$ and $x \in \bigoplus_{i=1}^{q}{}^{P} DIS_i(D_j)$, so $x \in \bigoplus_{i=1}^{q}{}^{P} DIS_i(D_i) \cap \bigoplus_{i=1}^{q}{}^{P} DIS_i(D_j)$. Known by the arbitrariness of x, $\forall D_i, D_j \in U/D$, $\bigoplus_{i=1}^{q}{}^{P} DIS_i(D_i \cap D_j) \subseteq \bigoplus_{i=1}^{q}{}^{P} DIS_i(D_i) \cap \bigoplus_{i=1}^{q}{}^{P} DIS_i(D_j)$.

(5) $\forall x \in \bigotimes_{i=1}^{q}{}^{P} DIS_i(D_i) \cup \bigotimes_{i=1}^{q}{}^{P} DIS_i(D_j)$, $x \in \bigotimes_{i=1}^{q}{}^{P} DIS_i(D_i)$ or $x \in \bigotimes_{i=1}^{q}{}^{P} DIS_i(D_j)$, $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x)}{q} = 1$ or $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} = 1$. What's more, $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i \cup D_j}^{DIS_i}(x)}{q} \geq \left(\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x)}{q} \vee \frac{\sum_{i=1}^{q} \mathcal{DR}_{D_j}^{DIS_i}(x)}{q} \right) = 1$, so $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i \cup D_j}^{DIS_i}(x)}{q} \geq 1$, $x \in \bigotimes_{i=1}^{q}{}^{P} DIS_i(D_i \cup D_j)$. Known by the arbitrariness of x, $\forall D_i, D_j \in U/D$, $\bigotimes_{i=1}^{q}{}^{P} DIS_i(D_i \cup D_j) \supseteq \bigotimes_{i=1}^{q}{}^{P} DIS_i(D_i) \cup \bigotimes_{i=1}^{q}{}^{P} DIS_i(D_j)$.

For any $x \in \bigoplus_{i=1}^{q}{}^{P} DIS_i(D_i \cup D_j) \Leftrightarrow \frac{\sum_{i=1}^{q} \mathcal{DR}_{D_i \cup D_j}^{DIS_i}(x)}{q} > 0 \Leftrightarrow \exists i \leq q$, s.t $[x]_{DIS_i} \cap (D_i \cup D_j) \neq \emptyset \Leftrightarrow \exists i \leq q$, $[x]_{DIS_i} \cap D_i \neq \emptyset$ or $[x]_{DIS_i} \cap D_j \neq \emptyset \Leftrightarrow \frac{\sum_{i=1}^{q} \mathcal{DR}_{D_i}^{DIS_i}(x)}{q} > 0$ or $\frac{\sum_{i=1}^{q} \mathcal{DR}_{D_i}^{DIS_i}(x)}{q} > 0 \Leftrightarrow x \in \bigoplus_{i=1}^{q}{}^{P} DIS_i(D_i) \cup \bigoplus_{i=1}^{q}{}^{P} DIS_i(D_j)$. Known by the arbitrariness of x, $\forall D_i, D_j \in U/D$, $\bigoplus_{i=1}^{q}{}^{P} DIS_i(D_i \cup D_j) = \bigoplus_{i=1}^{q}{}^{P} DIS_i(D_i) \cup \bigoplus_{i=1}^{q}{}^{P} DIS_i(D_j)$.

3.3. Optimistic fixed aggregation operator and optimistic possible aggregation operator of MsDIS

Like the previous subsection, the optimistic fixed aggregation operator and optimistic possible aggregation operator of MsDIS are constructed from the angle of optimistic multi-granulation rough set.

Definition 3.6. Let $M_SDIS = \{DIS_1, DIS_2, \ldots, DIS_q\}$ be a MsDIS, where $DIS_i = (U, AT, V_i, F_i, DT, D_i, G_i)$ $(i = 1, 2, \ldots, q)$. $U/D = \{D_1, D_2, \ldots, D_m\}$, for any $D_j \in U/D$ $(j = 1, 2, \ldots, m)$, the optimistic fixed aggregation operator and optimistic possible aggregation operator of MsDIS are defined as follows:

$$\bigotimes_{i=1}^{q}{}^{O} DIS_i(D_j) = \left\{x \in U \,\middle|\, \vee_{i=1}^{q} ([x]_{DIS_i} \subseteq D_j)\right\} = \left\{ x \in U \,\middle|\, \frac{\sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} > 0 \right\}, \tag{19}$$

$$\bigoplus_{i=1}^{q}{}^{O} DIS_i(D_j) = \left\{x \in U \,\middle|\, \wedge_{i=1}^{q} ([x]_{DIS_i} \cap D_j \neq \emptyset)\right\} = \left\{ x \in U \,\middle|\, \frac{\sum_{i=1}^{q} \mathcal{DR}_{D_j}^{DIS_i}(x)}{q} = 1 \right\}. \tag{20}$$

Definition 3.7. The definition of optimistic fusion approximate accuracy and optimistic fusion roughness for dividing U/D is shown as follows:

$$\varrho_{MSD}(U/D)^O = \frac{\sum_{D_j \in U/D} \left| \bigotimes_{i=1}^{q}{}^{O} DIS_i(D_j) \right|}{\sum_{D_j \in U/D} \left| \bigoplus_{i=1}^{q}{}^{O} DIS_i(D_j) \right|}, \quad \rho_{MSD}(U/D)^O = 1 - \varrho_{MSD}(U/D)^O. \tag{21}$$

Property 3.4. *Let $M_SDIS = \{DIS_1, DIS_2, \ldots, DIS_q\}$ be a MsDIS, where $DIS_i = (U, AT, V_i, F_i, DT, D_i, G_i)$ $(i = 1, 2, \ldots, q)$. $U/D = \{D_1, D_2, \ldots, D_m\}$, for any $D_j \in U/D$ $(j = 1, 2, \ldots, m)$, the following properties are true.*

(1) $\bigotimes_{i=1}^{q}{}^{O} DIS_i(D_j) \subseteq D_j \subseteq \bigoplus_{i=1}^{q}{}^{O} DIS_i(D_j)$;

(2) $\bigotimes_{i=1}^{q} {}^{O} DIS_i(\emptyset) = \bigoplus_{i=1}^{q} {}^{O} DIS_i(\emptyset) = \emptyset$;

(3) $\bigotimes_{i=1}^{q} {}^{O} DIS_i(U) = \bigoplus_{i=1}^{q} {}^{O} DIS_i(U) = U$;

(4) $\forall D_i, D_j \in U/D, \bigotimes_{i=1}^{q} {}^{O} DIS_i(D_i \cap D_j) \subseteq \bigotimes_{i=1}^{q} {}^{O} DIS_i(D_i) \cap \bigotimes_{i=1}^{q} {}^{O} DIS_i(D_j)$,

$\bigoplus_{i=1}^{q} {}^{O} DIS_i(D_i \cap D_j) \subseteq \bigoplus_{i=1}^{q} {}^{O} DIS_i(D_i) \cap \bigoplus_{i=1}^{q} {}^{O} DIS_i(D_j)$;

(5) $\forall D_i, D_j \in U/D, \bigotimes_{i=1}^{q} {}^{O} DIS_i(D_i \cup D_j) \supseteq \bigotimes_{i=1}^{q} {}^{O} DIS_i(D_i) \cup \bigotimes_{i=1}^{q} {}^{O} DIS_i(D_j)$,

$\bigoplus_{i=1}^{q} {}^{O} DIS_i(D_i \cup D_j) \supseteq \bigoplus_{i=1}^{q} {}^{O} DIS_i(D_i) \cup \bigoplus_{i=1}^{q} {}^{O} DIS_i(D_j)$.

Proof. (1) For every $x \in \bigotimes_{i=1}^{q} {}^{O} DIS_i(D_j)$, we can know $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} > 0$. So $\exists\ i \le q$, s.t $[x]_{DIS_i} \subseteq D_j$. Known by the arbitrariness of x, $\bigotimes_{i=1}^{q} {}^{O} DIS_i(D_j) \subseteq D_j$. For each $x \in D_j$, we can know $[x]_{DIS_i} \cap D_j \neq \emptyset$. So $\frac{\sum_{i=1}^{q} \mathcal{DR}_{D_j}^{DIS_i}(x)}{q} = 1$, $x \in \bigoplus_{i=1}^{q} {}^{O} DIS_i(D_j)$. Known by the arbitrariness of x, $D_j \subseteq \bigoplus_{i=1}^{q} {}^{O} DIS_i(D_j)$. Thus, the property (1) is proved.

(2) Through the definition of decision support characteristic function and decision related characteristic function, we can obtain $\mathcal{DS}_{\emptyset}^{DIS_i}(x) = 0, \mathcal{DR}_{\emptyset}^{DIS_i}(x) = 0$. So we can have the following two equations:

$$\bigotimes_{i=1}^{q} {}^{O} DIS_i(\emptyset) = \left\{ x \in U \,\middle|\, \frac{\sum_{i=1}^{q} \mathcal{DS}_{\emptyset}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} 0}{q} = 0 \right\} = \emptyset,$$

$$\bigoplus_{i=1}^{q} {}^{O} DIS_i(\emptyset) = \left\{ x \in U \,\middle|\, \frac{\sum_{i=1}^{q} \mathcal{DR}_{\emptyset}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} 0}{q} = 0 \right\} = \emptyset.$$

(3) Through the definition of decision support characteristic function and decision related characteristic function, we can obtain $\mathcal{DS}_{U}^{DIS_i}(x) = 1, \mathcal{DR}_{U}^{DIS_i}(x) = 1$. So we can gain the following two equations:

$$\bigotimes_{i=1}^{q} {}^{O} DIS_i(U) = \left\{ x \in U \,\middle|\, \frac{\sum_{i=1}^{q} \mathcal{DS}_{U}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} 1}{q} = 1 \right\} = U,$$

$$\bigoplus_{i=1}^{q} {}^{O} DIS_i(U) = \left\{ x \in U \,\middle|\, \frac{\sum_{i=1}^{q} \mathcal{DR}_{U}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} 1}{q} = 1 \right\} = U.$$

(4) For every $x \in \bigotimes_{i=1}^{q} {}^{O} DIS_i(D_i \cap D_j)$, we can have $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i \cap D_j}^{DIS_i}(x)}{q} = \frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x) \wedge \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} > 0$. Because of

$$\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x) \wedge \sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} \ge \frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x) \wedge \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} > 0,$$

Table 1
A multi-source decision information system.

	DIS_1				DIS_2				DIS_3				DIS_4				
	a_1	a_2	a_3	a_4	a_1	a_2	a_3	a_4	a_1	a_2	a_3	a_4	a_1	a_2	a_3	a_4	d
o_1	1	2	2	1	1	2	2	1	1	2	1	1	1	2	2	1	1
o_2	1	2	1	1	1	2	2	1	1	2	1	1	1	2	1	1	1
o_3	1	1	2	1	1	1	2	1	1	1	1	1	1	1	2	1	0
o_4	0	1	1	1	1	1	1	1	0	1	2	1	0	1	2	0	1
o_5	2	1	1	2	0	1	1	1	1	1	1	1	2	2	1	1	0
o_6	0	1	1	0	0	1	2	0	0	1	1	0	1	1	2	0	1
o_7	1	1	2	1	2	2	2	1	1	2	1	1	1	2	1	1	0
o_8	1	1	1	0	2	1	1	0	1	1	1	0	1	1	1	0	1
o_9	2	1	1	0	2	1	1	1	2	1	2	1	2	1	2	1	0
o_{10}	1	1	1	0	1	1	1	1	0	1	2	1	0	1	2	0	0

thus $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x)}{q} > 0$ and $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} > 0$, i.e. $x \in \bigotimes_{i=1}^{q}{}^{O} DIS_i(D_i)$ and $x \in \bigotimes_{i=1}^{q}{}^{O} DIS_i(D_j)$. So $x \in \bigotimes_{i=1}^{q}{}^{O} DIS_i(D_i) \cap \bigotimes_{i=1}^{q}{}^{O} DIS_i(D_j)$. Known by the arbitrariness of x, $\forall D_i, D_j \in U/D$, $\bigotimes_{i=1}^{q}{}^{O} DIS_i(D_i \cap D_j) \subseteq \bigotimes_{i=1}^{q}{}^{O} DIS_i(D_i) \cap \bigotimes_{i=1}^{q}{}^{O} DIS_i(D_j)$. Similarly, we can prove $\bigoplus_{i=1}^{q}{}^{O} DIS_i(D_i \cap D_j) \subseteq \bigoplus_{i=1}^{q}{}^{O} DIS_i(D_i) \cap \bigoplus_{i=1}^{q}{}^{O} DIS_i(D_j)$.

(5) For any $x \in \bigotimes_{i=1}^{q}{}^{O} DIS_i(D_i) \cup \bigotimes_{i=1}^{q}{}^{O} DIS_i(D_j)$, $x \in \bigotimes_{i=1}^{q}{}^{O} DIS_i(D_i)$ or $x \in \bigotimes_{i=1}^{q}{}^{O} DIS_i(D_j)$, $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x)}{q} > 0$ or $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_j}^{DIS_i}(x)}{q} > 0$. What's more, $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i \cup D_j}^{DIS_i}(x)}{q} \geq \left(\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i}^{DIS_i}(x)}{q} \vee \frac{\sum_{i=1}^{q} \mathcal{DR}_{D_j}^{DIS_i}(x)}{q} \right) > 0$, so $\frac{\sum_{i=1}^{q} \mathcal{DS}_{D_i \cup D_j}^{DIS_i}(x)}{q} > 0$, $x \in \bigotimes_{i=1}^{q}{}^{O} DIS_i(D_i \cup D_j)$. Known by the arbitrariness of x, $\forall D_i, D_j \in U/D$, $\bigotimes_{i=1}^{q}{}^{O} DIS_i(D_i \cup D_j) \supseteq \bigotimes_{i=1}^{q}{}^{O} DIS_i(D_i) \cup \bigotimes_{i=1}^{q}{}^{O} DIS_i(D_j)$. Similarly, we can prove $\bigoplus_{i=1}^{q}{}^{O} DIS_i(D_i \cup D_j) \supseteq \bigoplus_{i=1}^{q}{}^{O} DIS_i(D_i) \cup \bigoplus_{i=1}^{q}{}^{O} DIS_i(D_j)$.

Property 3.5. *Let $M_S DIS = \{DIS_1, DIS_2, \ldots, DIS_q\}$ be a MsDIS, where $DIS_i = (U, AT, V_i, F_i, DT, D_i, G_i)$ $(i = 1, 2, \ldots, q)$. α, β are two thresholds which satisfied $\alpha + \beta = 1$ and $\alpha \in (0, 1]$, $\beta \in [0, 1)$. $U/D = \{D_1, D_2, \ldots, D_m\}$, three kinds of fusion approximation accuracy (Eq. (15), Eq. (18), Eq. (21)) satisfies the following properties.*

$$\varrho_{MSD}(U/D)^P \leq \varrho_{MSD}(U/D)_{(\alpha,\beta)} \leq \varrho_{MSD}(U/D)^O. \tag{22}$$

Proof. It's easy to prove by Definition 3.2-3.7 and Property 3.2.

3.4. Case study

There are 10 plots of land that need to be tested to determine whether they are suitable for growing crops. If only the soil of a certain point in a piece of land is selected for testing, the conclusion may be greatly deviated. So we chose to collect the soil in four directions of each piece of land for testing. Finally, experts in the industry will judge whether the land is suitable for growing crops by neutralizing the results of the four directions. We use the soil in four directions as four information sources to obtain a MsDIS. In this MsDIS, our research object is $U = \{o_1, o_2, \ldots, o_{10}\}$, each element in U represents a piece of land. The elements in $AT = \{a_1, a_2, a_3, a_4\}$ represent four different microelement in soil, $DT = \{d\}$ is the decision attribute, the role here is to determine whether the land is suitable for growing crops. If $d = 1$, the land is suitable for planting crops; if $d = 0$, the land is not suitable for planting crops. More detailed information about this MsDIS is shown in Table 1.

First we should calculate the equivalence class of each object under each source.

The equivalence class of the object in DIS_1:

$[o_1]_{DIS_1} = \{o_1\}$, $[o_2]_{DIS_1} = \{o_2\}$, $[o_3]_{DIS_1} = \{o_3, o_7\}$, $[o_4]_{DIS_1} = \{o_4\}$, $[o_5]_{DIS_1} = \{o_5\}$, $[o_6]_{DIS_1} = \{o_6\}$, $[o_7]_{DIS_1} = \{o_3, o_7\}$, $[o_8]_{DIS_1} = \{o_8, o_{10}\}$, $[o_9]_{DIS_1} = \{o_9\}$, $[o_{10}]_{DIS_1} = \{o_8, o_{10}\}$.

The equivalence class of the object in DIS_2:

$[o_1]_{DIS_2} = \{o_1, o_2\}$, $[o_2]_{DIS_2} = \{o_1, o_2\}$, $[o_3]_{DIS_2} = \{o_3\}$, $[o_4]_{DIS_2} = \{o_4, o_{10}\}$, $[o_5]_{DIS_2} = \{o_5\}$, $[o_6]_{DIS_2} = \{o_6\}$, $[o_7]_{DIS_2} = \{o_7\}$, $[o_8]_{DIS_2} = \{o_8\}$, $[o_9]_{DIS_2} = \{o_9\}$, $[o_{10}]_{DIS_2} = \{o_4, o_{10}\}$.

The equivalence class of the object in DIS_3:
$[o_1]_{DIS_3} = \{o_1, o_2, o_7\}$, $[o_2]_{DIS_3} = \{o_1, o_2, o_7\}$, $[o_3]_{DIS_3} = \{o_3, o_5\}$, $[o_4]_{DIS_3} = \{o_4, o_{10}\}$, $[o_5]_{DIS_3} = \{o_3, o_5\}$, $[o_6]_{DIS_3} = \{o_6\}$, $[o_7]_{DIS_3} = \{o_1, o_2, o_7\}$, $[o_8]_{DIS_3} = \{o_8\}$, $[o_9]_{DIS_3} = \{o_9\}$, $[o_{10}]_{DIS_3} = \{o_4, o_{10}\}$.
The equivalence class of the object in DIS_4:
$[o_1]_{DIS_4} = \{o_1\}$, $[o_2]_{DIS_4} = \{o_2, o_7\}$, $[o_3]_{DIS_4} = \{o_3\}$, $[o_4]_{DIS_4} = \{o_4, o_{10}\}$, $[o_5]_{DIS_4} = \{o_5\}$, $[o_6]_{DIS_4} = \{o_6\}$, $[o_7]_{DIS_4} = \{o_2, o_7\}$, $[o_8]_{DIS_4} = \{o_8\}$, $[o_9]_{DIS_4} = \{o_9\}$, $[o_{10}]_{DIS_4} = \{o_4, o_{10}\}$.
Division under decision attribute d: $U/D = \{\{o_1, o_2, o_4, o_6, o_8\}, \{o_3, o_5, o_7, o_9, o_{10}\}\}$.

Then calculate the decision support characteristic function and decision related characteristic function of the object for each decision class under each source:
The decision support characteristic function and decision related characteristic function for $D_1 = \{o_1, o_2, o_4, o_6, o_8\}$ are shown as follows:

$$\begin{aligned}
&\mathcal{DS}_{D_1}^{DIS_1}(o_1) = 1,\ \mathcal{DS}_{D_1}^{DIS_2}(o_1) = 1,\ \mathcal{DS}_{D_1}^{DIS_3}(o_1) = 0,\ \mathcal{DS}_{D_1}^{DIS_4}(o_1) = 1.\ \mathcal{DS}_{D_1}^{DIS_1}(o_2) = 1,\ \mathcal{DS}_{D_1}^{DIS_2}(o_2) = 1,\\
&\mathcal{DS}_{D_1}^{DIS_3}(o_2) = 0,\ \mathcal{DS}_{D_1}^{DIS_4}(o_2) = 0.\ \mathcal{DS}_{D_1}^{DIS_1}(o_3) = 0,\ \mathcal{DS}_{D_1}^{DIS_2}(o_3) = 0,\ \mathcal{DS}_{D_1}^{DIS_3}(o_3) = 0,\ \mathcal{DS}_{D_1}^{DIS_4}(o_3) = 0.\\
&\mathcal{DS}_{D_1}^{DIS_1}(o_4) = 1,\ \mathcal{DS}_{D_1}^{DIS_2}(o_4) = 0,\ \mathcal{DS}_{D_1}^{DIS_3}(o_4) = 0,\ \mathcal{DS}_{D_1}^{DIS_4}(o_4) = 0.\ \mathcal{DS}_{D_1}^{DIS_1}(o_5) = 0,\ \mathcal{DS}_{D_1}^{DIS_2}(o_5) = 0,\\
&\mathcal{DS}_{D_1}^{DIS_3}(o_5) = 0,\ \mathcal{DS}_{D_1}^{DIS_4}(o_5) = 0.\ \mathcal{DS}_{D_1}^{DIS_1}(o_6) = 1,\ \mathcal{DS}_{D_1}^{DIS_2}(o_6) = 1,\ \mathcal{DS}_{D_1}^{DIS_3}(o_6) = 1,\ \mathcal{DS}_{D_1}^{DIS_4}(o_6) = 1.\\
&\mathcal{DS}_{D_1}^{DIS_1}(o_7) = 0,\ \mathcal{DS}_{D_1}^{DIS_2}(o_7) = 0,\ \mathcal{DS}_{D_1}^{DIS_3}(o_7) = 0,\ \mathcal{DS}_{D_1}^{DIS_4}(o_7) = 0.\ \mathcal{DS}_{D_1}^{DIS_1}(o_8) = 0,\ \mathcal{DS}_{D_1}^{DIS_2}(o_8) = 1,\\
&\mathcal{DR}_{D_1}^{DIS_3}(o_8) = 1,\ \mathcal{DR}_{D_1}^{DIS_4}(o_8) = 1.\ \mathcal{DR}_{D_1}^{DIS_1}(o_9) = 0,\ \mathcal{DR}_{D_1}^{DIS_2}(o_9) = 0,\ \mathcal{DR}_{D_1}^{DIS_3}(o_9) = 0,\ \mathcal{DR}_{D_1}^{DIS_4}(o_9) = 0.\\
&\mathcal{DR}_{D_1}^{DIS_1}(o_{10}) = 1,\ \mathcal{DR}_{D_1}^{DIS_2}(o_{10}) = 1,\ \mathcal{DR}_{D_1}^{DIS_3}(o_{10}) = 1,\ \mathcal{DR}_{D_1}^{DIS_4}(o_{10}) = 1.
\end{aligned}$$

The decision support characteristic function and decision related characteristic function for $D_2 = \{o_3, o_5, o_7, o_9, o_{10}\}$ are shown as follows:

$$\begin{aligned}
&\mathcal{DS}_{D_2}^{DIS_1}(o_1) = 0,\ \mathcal{DS}_{D_2}^{DIS_2}(o_1) = 0,\ \mathcal{DS}_{D_2}^{DIS_3}(o_1) = 0,\ \mathcal{DS}_{D_2}^{DIS_4}(o_1) = 0.\ \mathcal{DS}_{D_2}^{DIS_1}(o_2) = 0,\ \mathcal{DS}_{D_2}^{DIS_2}(o_2) = 0,\\
&\mathcal{DS}_{D_2}^{DIS_3}(o_2) = 0,\ \mathcal{DS}_{D_2}^{DIS_4}(o_2) = 0.\ \mathcal{DS}_{D_2}^{DIS_1}(o_3) = 1,\ \mathcal{DS}_{D_2}^{DIS_2}(o_3) = 1,\ \mathcal{DS}_{D_2}^{DIS_3}(o_3) = 1,\ \mathcal{DS}_{D_2}^{DIS_4}(o_3) = 1.\\
&\mathcal{DS}_{D_2}^{DIS_1}(o_4) = 0,\ \mathcal{DS}_{D_2}^{DIS_2}(o_4) = 0,\ \mathcal{DS}_{D_2}^{DIS_3}(o_4) = 0,\ \mathcal{DS}_{D_2}^{DIS_4}(o_4) = 0.\ \mathcal{DS}_{D_2}^{DIS_1}(o_5) = 1,\ \mathcal{DS}_{D_2}^{DIS_2}(o_5) = 1,\\
&\mathcal{DS}_{D_2}^{DIS_3}(o_5) = 1,\ \mathcal{DS}_{D_2}^{DIS_4}(o_5) = 1.\ \mathcal{DS}_{D_2}^{DIS_1}(o_6) = 0,\ \mathcal{DS}_{D_2}^{DIS_2}(o_6) = 0,\ \mathcal{DS}_{D_2}^{DIS_3}(o_6) = 0,\ \mathcal{DS}_{D_2}^{DIS_4}(o_6) = 0.\\
&\mathcal{DS}_{D_2}^{DIS_1}(o_7) = 1,\ \mathcal{DS}_{D_2}^{DIS_2}(o_7) = 1,\ \mathcal{DS}_{D_2}^{DIS_3}(o_7) = 0,\ \mathcal{DS}_{D_2}^{DIS_4}(o_7) = 0.\ \mathcal{DS}_{D_2}^{DIS_1}(o_8) = 0,\ \mathcal{DS}_{D_2}^{DIS_2}(o_8) = 0,\\
&\mathcal{DS}_{D_2}^{DIS_3}(o_8) = 0,\ \mathcal{DS}_{D_2}^{DIS_4}(o_8) = 0.\ \mathcal{DR}_{D_2}^{DIS_1}(o_9) = 1,\ \mathcal{DR}_{D_2}^{DIS_2}(o_9) = 1,\ \mathcal{DR}_{D_2}^{DIS_3}(o_9) = 1,\ \mathcal{DR}_{D_2}^{DIS_4}(o_9) = 1.\\
&\mathcal{DR}_{D_2}^{DIS_1}(o_{10}) = 1,\ \mathcal{DR}_{D_2}^{DIS_2}(o_{10}) = 1,\ \mathcal{DR}_{D_2}^{DIS_3}(o_{10}) = 1,\ \mathcal{DR}_{D_2}^{DIS_4}(o_{10}) = 1.
\end{aligned}$$

In MsDIS, let $\alpha = 0.75$, then fusion results are calculated by the aggregation operators:

$$\bigotimes_{i=1}^{4} DIS_i(D_1)_{0.75} = \{o_1, o_6, o_8\},\ \bigoplus_{i=1}^{4} DIS_i(D_1)_{0.25} = \{o_1, o_2, o_4, o_6, o_7, o_8, o_{10}\},$$
$$\bigotimes_{i=1}^{4} DIS_i(D_2)_{0.75} = \{o_3, o_5, o_9\},\ \bigoplus_{i=1}^{4} DIS_i(D_2)_{0.25} = \{o_2, o_3, o_4, o_5, o_7, o_9, o_{10}\}.$$

The fusion approximate accuracy and fusion roughness are shown as follows:

$$\varrho_{MSD}(U/D)_{(0.75,0.25)} = \frac{\sum_{D_j \in U/D} \left| \bigotimes_{i=1}^{4} DIS_i(D_j)_{0.75} \right|}{\sum_{D_i \in U/D} \left| \bigoplus_{i=1}^{4} DIS_i(D_j)_{0.25} \right|} = \frac{6}{14},$$
$$\rho_{MSD}(U/D)_{(0.75,0.25)} = 1 - \varrho_{MSD}(U/D)_{(0.75,0.25)} = \frac{8}{14}.$$

In MsDIS, fusion results are calculated by the optimism aggregation operators:

$$\bigotimes_{i=1}^{4}{}^{O} DIS_i(D_1) = \{o_1, o_2, o_4, o_6, o_8\},\ \bigoplus_{i=1}^{4}{}^{O} DIS_i(D_1) = \{o_1, o_2, o_4, o_6, o_8, o_{10}\},$$
$$\bigotimes_{i=1}^{4}{}^{O} DIS_i(D_2) = \{o_3, o_5, o_7, o_9\},\ \bigoplus_{i=1}^{4}{}^{O} DIS_i(D_2) = \{o_3, o_5, o_7, o_9, o_{10}\}.$$

The fusion approximate accuracy and fusion roughness are shown as follows:

$$\varrho_{MSD}(U/D)^O = \frac{\sum_{D_j \in U/D} \left| \bigotimes_{i=1}^{4}{}^{O} DIS_i(D_j) \right|}{\sum_{D_i \in U/D} \left| \bigoplus_{i=1}^{4}{}^{O} DIS_i(D_j) \right|} = \frac{9}{11}, \rho_{MSD}(U/D)^O = 1 - \varrho_{MSD}(U/D)^O = \frac{2}{11}.$$

In MsDIS, fusion results are calculated by the pessimistic aggregation operators:

$$\bigotimes_{i=1}^{4}{}^{P} DIS_i(D_1) = \{o_6\}, \bigoplus_{i=1}^{4}{}^{P} DIS_i(D_1) = \{o_1, o_2, o_4, o_6, o_7, o_8, o_{10}\},$$

$$\bigotimes_{i=1}^{4}{}^{P} DIS_i(D_2) = \{o_3, o_5, o_9\}, \bigoplus_{i=1}^{4}{}^{P} DIS_i(D_2) = \{o_1, o_2, o_3, o_4, o_5, o_7, o_8, o_9, o_{10}\}.$$

The fusion approximate accuracy and fusion roughness are shown as follows:

$$\varrho_{MSD}(U/D)^P = \frac{\sum_{D_j \in U/D} \left| \bigotimes_{i=1}^{4}{}^{P} DIS_i(D_j) \right|}{\sum_{D_i \in U/D} \left| \bigoplus_{i=1}^{4}{}^{P} DIS_i(D_j) \right|} = \frac{1}{4}, \rho_{MSD}(U/D)^P = 1 - \varrho_{MSD}(U/D)^P = \frac{3}{4}.$$

Obviously, $\varrho_{MSD}(U/D)^P \le \varrho_{MSD}(U/D)_{(0.75,0.25)} \le \varrho_{MSD}(U/D)^O$.

4. Experimental analysis

In the third part of this paper, we proposed three groups of aggregation operators in order to directly integrate MsDIS. In this section, we prove the effectiveness and feasibility of the aggregation operators through a series of experiments. Firstly, we propose an algorithm (Algorithm 1) for calculating the fusion approximation accuracy of MsDIS. Meanwhile, we design Algorithm 2 to calculate the approximate accuracy of mean fusion in order to verify the effectiveness of the aggregation operators. Secondly, we download 9 data sets on UCI (http://archive.ics.uci.edu/ml/datasets.html). The specific information of the data sets is shown in Table 2. The entire experiment was run on a private computer. The specific operating environment, including hardware and software, is shown in Table 3. Finally, we did two groups of comparative experiments.

In order to make readers understand the mean fusion method mentioned in Algorithm 2 better, next we introduce the steps of mean fusion method.

The method of mean fusion: Given a $M_S DIS = \{DIS_1, DIS_2, \dots, DIS_q\}$. Firstly, the MsDIS is integrated into single-source decision information system (SsDIS) through

$$NewDIS(x_j, a_k) = fix \frac{\sum_{i=1}^{q} DIS_i(x_j, a_k)}{q}, \tag{23}$$

where $NewDIS(x_j, a_k)$ represents the fused information and $DIS_i(x_j, a_k)$ represents the initial information under the ith source. Then, the upper and lower approximation sets are found from SsDIS. Finally, the Approximation accuracy can be computed.

We analyze the time complexity of the two algorithms as shown below. In Algorithm 1, from step 2 to step 14, we calculate the sum of the decision support characteristic function and the sum of the decision related characteristic function of each object in each decision class, and the time complexity is $\mathcal{O}(m \times n \times q)$; from step 15 to step 25, we fuse the MsDIS through the aggregate operator constructed in section 3, and the time complexity is $\mathcal{O}(m \times n)$. In Algorithm 2, from step 2 to step 10, we perform mean fusion on MsDIS, and the time complexity is $\mathcal{O}(n \times p \times q)$; from step 11 to step 21, we compute the upper and lower approximations of each decision class, and the time complexity is $\mathcal{O}(m \times n)$.

It is well known that it is not easy to directly obtain a MsDIS from a machine learning database. Therefore, we use the method of adding white noise and random noise into the original data set to obtain the data required for the experiment. Next, we introduce the method of adding white noise into the original data. First, generate q numbers $(n_1, n_2, \dots, n_q)$ that satisfy the normal distribution. Add white noise as follows:

Algorithm 1: Calculating the fusion approximation accuracy of MsDIS.

Input : (α, β), $M_S DIS = \{DIS_1, DIS_2, \ldots, DIS_q\}$, $U/D = \{D_1, D_2, \ldots, D_m\}$
Output : The fusion approximation accuracy of U/D

begin
  for $k = 1 : m$ **do** /* the m is the number of decision classes */
    for $i = 1 : n$ **do** /* the n is the number of objects */
      $\mathcal{DS}(i) = 0; \mathcal{DR}(i) = 0;$
      for $j = 1 : q$ **do** /* the q is the number of information sources */
        if $[x_i]_{DIS_j} \subseteq D_k$ **then**
          $\mathcal{DS}(i) = \mathcal{DS}(i) + 1;$
        end
        if $[x_i]_{DIS_j} \cap D_k \neq \emptyset$ **then**
          $\mathcal{DR}(i) = \mathcal{DR}(i) + 1;$
        end
      end
    end
  end
  for $k = 1 : m$ **do** /* the m is the number of decision classes */
    $\bigotimes_{j=1}^{q} DIS_j(D_k)_\alpha \leftarrow \emptyset; \bigoplus_{j=1}^{q} DIS_j(D_k)_\beta \leftarrow \emptyset;$
    for $i = 1 : n$ **do**
      if $\frac{\mathcal{DS}(i)}{q} \geq \alpha$ **then**
        $\bigotimes_{j=1}^{q} DIS_j(D_k)_\alpha = \bigotimes_{j=1}^{q} DIS_j(D_k)_\alpha \cup \{x_i\};$
      end
      if $\frac{\mathcal{DR}(i)}{q} > \beta$ **then**
        $\bigoplus_{j=1}^{q} DIS_j(D_k)_\beta = \bigoplus_{j=1}^{q} DIS_j(D_k)_\beta \cup \{x_i\};$
      end
    end
  end
  return : $\varrho_{MSD}(U/D)_{(\alpha,\beta)} = \frac{\sum_{k=1}^{m} \left| \bigotimes_{j=1}^{q} DIS_j(D_k)_\alpha \right|}{\sum_{k=1}^{m} \left| \bigoplus_{j=1}^{q} DIS_j(D_k)_\beta \right|};$
end

$$IS_i(x, a) = \begin{cases} IS(x, a) + n_i, & if \quad 0 \leq |n_i| \leq 1, \\ IS(x, a), & otherwise. \end{cases} \tag{24}$$

Similarly, the method of adding random noise is as follows:

$$IS_i(x, a) = \begin{cases} IS(x, a) + r_i, & if \quad 0 \leq |r_i| \leq 1, \\ IS(x, a), & otherwise. \end{cases} \tag{25}$$

In the original information system, the value of object x under attribute a is denoted by us as $IS(x, a)$. After adding noise, the corresponding value in the ith information system is recorded as $IS_i(x, a)$. Then we randomly select 40% of the data in the original information system to add white noise, the remaining 20% add random noise, and the rest keep the information of the original system unchanged. In the experiment of this paper, we construct 10 information sources in each original data set by adding noise. Finally, we obtain a multi-source decision information system.

4.1. Comparison of fusion approximation accuracy and approximation accuracy after mean fusion

In this subsection, we compare pessimistic aggregation operators, optimistic aggregation operators and (α, β)-aggregation operators with mean fusion by approximate accuracy. Fig. 3 shows the approximation accuracy of pessimistic fusion, optimistic fusion, mean fusion and (α, β)-fusion. The following conclusions can be drawn from the experimental results.

- In each data set, the fusion effect of the optimistic aggregation operators is the best, and the fusion effect of the pessimistic aggregation operators is the worst. For the (α, β)-aggregation operators, we can adjust the fusion result by adjusting the threshold (α, β), so for practical applications, the (α, β)-aggregation operators is more suitable for life.

Algorithm 2: Calculating the approximate accuracy of MsDIS after mean fusion.

Input : $M_S DIS = \{DIS_1, DIS_2, \ldots, DIS_q\}$, $U/D = \{D_1, D_2, \ldots, D_m\}$
Output : The fusion approximation accuracy of U/D

begin
  for $i = 1 : n$ **do** /* the n is the number of objects */
    for $k = 1 : p$ **do** /* the p is the number of attributes */
      $Sum(x_i, a_k) \leftarrow 0$;
      for $j = 1 : q$ **do** /* the q is the number of information sources */
        $Sum(x_i, a_k) \leftarrow Sum(x_i, a_k) + DIS_j(x_i, a_k)$;
      end
      $NIS(x_i, a_k) \leftarrow fix\frac{Sum(x_i, a_k)}{q}$;
    end
  end
  for $l = 1 : m$ **do** /* the m is the number of decision classes */
    $\underline{NIS}(D_l) \leftarrow \emptyset$; $\overline{NIS}(D_l) \leftarrow \emptyset$;
    for $i = 1 : n$ **do**
      if $[x_i]_{NIS} \subseteq D_l$ **then**
        $\underline{NIS}(D_l) = \underline{NIS}(D_l) \cup \{x_i\}$;
      end
      if $[x_i]_{NIS} \cap D_l \neq \emptyset$ **then**
        $\overline{NIS}(D_l) = \overline{NIS}(D_l) \cup \{x_i\}$;
      end
    end
  end
  return : $\varrho_{NIS}(U/D) = \frac{\sum_{l=1}^{m} \left|\underline{NIS}(D_l)\right|}{\sum_{l=1}^{m} \left|\overline{NIS}(D_l)\right|}$;
end

Table 2
Specific information about the data sets.

No.	Dataset name	Abbreviation	Objects	Attributes	Decision classes
1	Hayes-Roth	HR	160	4	3
2	Liver Disorders	LD	345	7	2
3	Balance Scale	BS	625	4	3
4	Banknote authentication	BA	1372	5	2
5	Winequality-Red	WR	1599	11	6
6	Wireless Indoor Localization	WIL	2000	7	4
7	Abalone	A	4177	8	3
8	Winequality-White	WW	4898	11	7
9	Nursery	N	12960	8	5

Table 3
Specific information about the operating environment.

Name	Model	Parameter
CPU	Intel(R) Core(TM) i3-2370M	2.40 GHz
Platform	MATLAB	R2016b
System	Windows 7	64 bit
Memory	DDR3	4 GB;1600 Mhz
Hard Disk	MQ01ABD050	500 GB

Moreover, as shown in Fig. 3, in each data set, we can find such a set of thresholds that its fusion effect is better than the mean fusion effect.

- In the data sets named “Hayes-roth”, “Balance Scale”, “Banknote authentication” and “Nursery”, when the values of the threshold α are taken to 0.6, 0.7, 0.8, and 0.6, respectively, the fusion effect is still better than that of the mean fusion. This shows that the gap between the different information sources in these four data sets is not particularly large, and the decision results are relatively consistent. Conversely, in the data sets named “Liver Disorders”, “Winequality-Red”, “Wireless Indoor Localization”, “Abalone” and “Winequality-White”, threshold α should not exceed 0.3, 0.3, 0.2, 0.2 and 0.3 respectively if the fusion effect is better than that of mean fusion. This means that the gaps between the various sources in these five data sets are large, and the decision results are quite different.

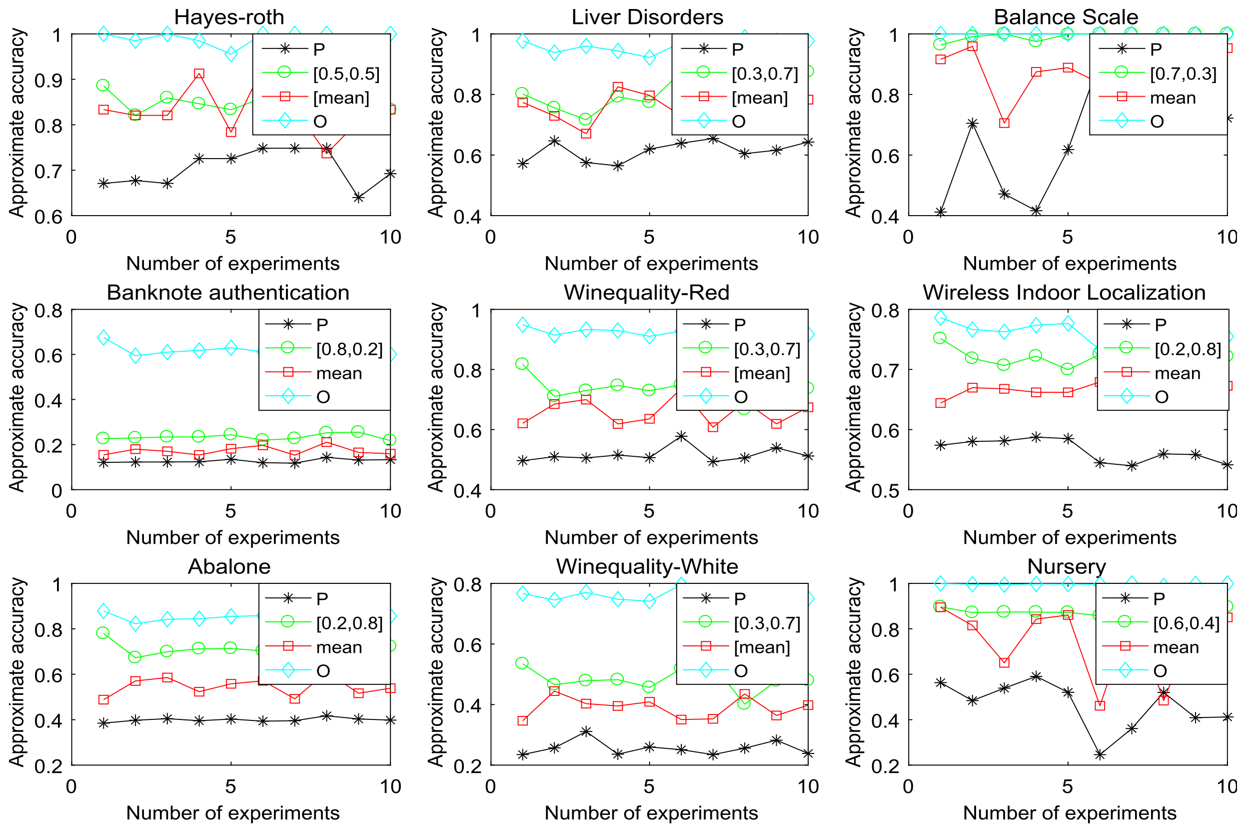

Fig. 3. Comparison of fusion approximation accuracy and approximation accuracy after mean fusion.

4.2. Comparison of fusion approximation accuracy under different thresholds

We set three groups thresholds for comparison with pessimistic and optimistic. The three groups thresholds are (0.4,0.6), (0.5,0.5) and (0.6,0.4), respectively. We calculate the fusion approximation accuracy under these three groups thresholds, and then compare it with the pessimistic fusion approximation accuracy and the optimistic fusion approximation accuracy. It can be seen from the Fig. 4: the fusion approximation accuracy of the optimistic aggregation operators is the largest, and the fusion approximation accuracy the pessimistic aggregation operators is the smallest. The fusion approximation accuracy of the (α, β)-aggregation operators is between the pessimistic approximation accuracy and the optimistic approximation accuracy, and decreases with the increase of threshold α. Obviously, the experimental results satisfy the Property 3.5.

5. Conclusions

In this paper, a novel fusion method of MsDIS is proposed from the perspective of multi-granulation. Firstly, the decision support characteristic function and the decision related characteristic function are constructed. Secondly, aggregation operators (including pessimistic aggregation operators and optimistic aggregation operators) induced by two functions are defined and the related properties of aggregation operators are researched. Meanwhile, for different aggregation operators, different fusion approximation precision is defined. These different fusion approximation precision provide measurement for our experiments. Finally, it is proved by experiments that the fusion method proposed by us has practicality when the gap between the sources is small. In the future, we will further study the fusion methods of heterogeneous multi-source information systems.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

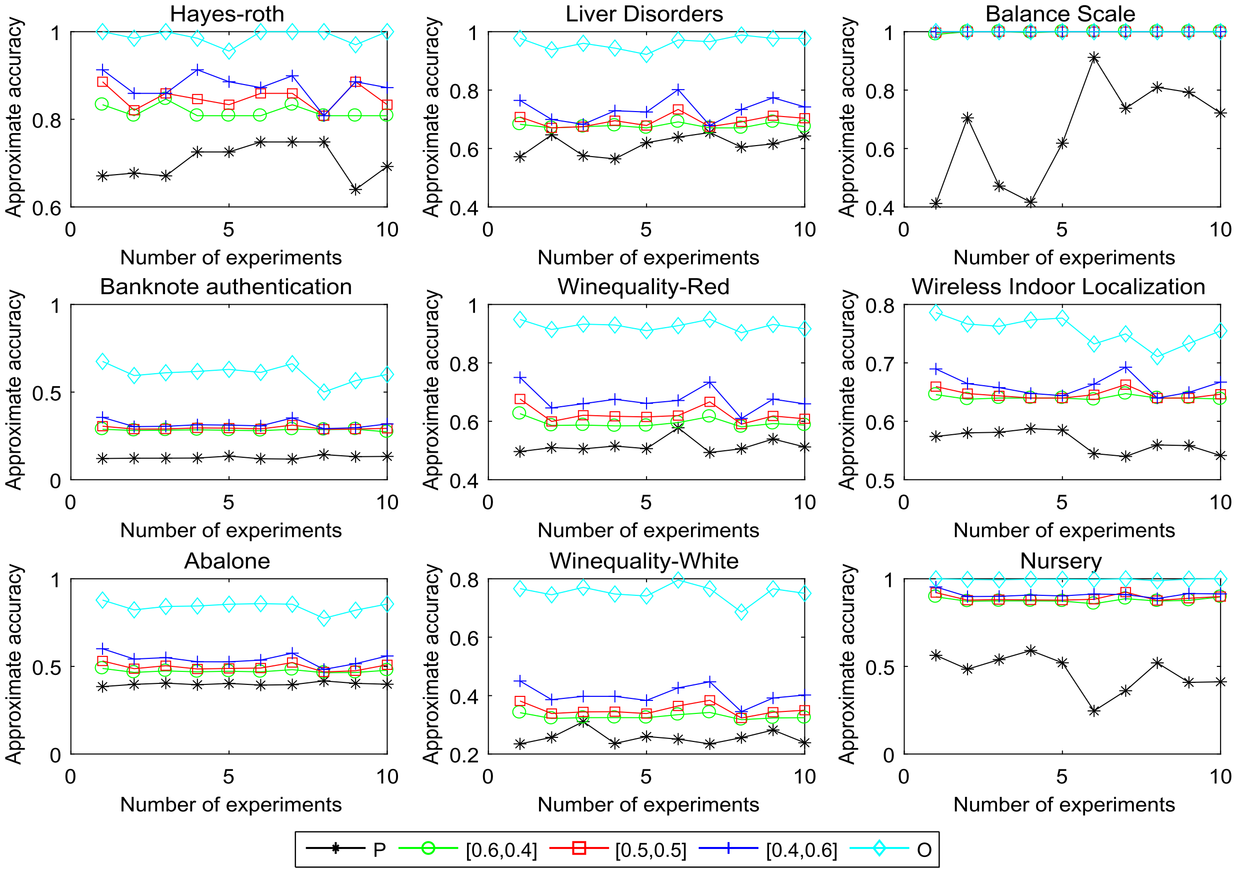

Fig. 4. Comparison of fusion approximation accuracy under different thresholds.

Acknowledgements

This paper is supported by the National Natural Science Foundation of China (No. 61976245, No. 61472463, No. 61772002), the Fundamental Research Funds for the Central Universities (No. XDJK2019B029), Graduate Student Innovation Program of Chongqing University of Technology (No. ycx20192080) and Graduate Innovation Foundation of CQ (No. CYS19323).

References

[1] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (5) (1982) 341–356.
[2] Z. Pawlak, Rough Sets: Theoretical Aspects of Reasoning About Data, Kluwer Academic Publishers, Norwell, MA, USA, 1992.
[3] Z. Pawlak, Rough set theory and its applications to data analysis, J. Cybern. 29 (7) (1998) 661–688.
[4] Y.H. Qian, J.Y. Liang, W. Pedrycz, C.Y. Dang, Positive approximation: an accelerator for attribute reduction in rough set theory, Artif. Intell. 174 (9–10) (2010) 597–618.
[5] W. Zhu, F.Y. Wang, Reduction and axiomization of covering generalized rough sets, Inf. Sci. 152 (1) (2003) 217–230.
[6] D.W. Pei, A generalized model of fuzzy rough sets, Int. J. Gen. Syst. 34 (5) (2005) 603–613.
[7] W.Z. Wu, J.S. Mi, W.X. Zhang, Generalized fuzzy rough sets, Inf. Sci. 151 (3) (2003) 263–282.
[8] Y.T. Guo, E.C.C. Tsang, W.H. Xu, D.G. Chen, Local logical disjunction double-quantitative rough sets, Inf. Sci. 500 (2019) 87–112.
[9] W.H. Xu, Y.T. Guo, Generalized multigranulation double-quantitative decision-theoretic rough set, Knowl.-Based Syst. 105 (2016) 190–205.
[10] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. Gen. Syst. 17 (2–3) (2007) 191–209.
[11] C. Cornelis, M.D. Cock, E.E. Kerre, Intuitionistic fuzzy rough sets: at the crossroads of imperfect knowledge, Expert Syst. 20 (5) (2010) 260–270.
[12] H.Y. Zhang, S.Y. Yang, Three-way group decisions with interval-valued decision-theoretic rough sets based on aggregating inclusion measures, Int. J. Approx. Reason. 110 (2019) 31–45.
[13] W.H. Xu, W.T. Li, Granular computing approach to two-way learning based on formal concept analysis in fuzzy datasets, IEEE Trans. Cybern. 46 (2) (2016) 366–379.
[14] Y.Y. Yao, Three-way decisions and granular computing, Int. J. Approx. Reason. 103 (2018) 107–123.
[15] Y.H. Qian, J.Y. Liang, C.Y. Dang, Incomplete multigranulation rough set, IEEE Trans. Syst. Man Cybern., Part A, Syst. Hum. 40 (2) (2010) 420–431.
[16] Y.H. Qian, J.Y. Liang, Y.Y. Yao, C.Y. Dang, MGRS: a multi-granulation rough set, Inf. Sci. Int. J. 180 (6) (2010) 949–970.
[17] W.H. Xu, W.X. Sun, X.Y. Zhang, W.X. Zhang, Multiple granulation rough set approach to ordered information systems, Int. J. Gen. Syst. 41 (5) (2012) 475–501.
[18] Q.H. Hu, J.S. Mi, D.G. Chen, Granular computing based machine learning in the era of big data: editorial, Inf. Sci. 378 (2017) 242–243.

[19] Y.H. Qian, X.Y. Liang, G.P. Lin, Q. Guo, J.Y. Liang, Local multigranulation decision-theoretic rough sets, Int. J. Approx. Reason. 82 (2017) 119–137.
[20] A.H. Tan, W.Z. Wu, J.J. Li, G.P. Lin, Evidence-theory-based numerical characterization of multigranulation rough sets in incomplete information systems, Fuzzy Sets Syst. 294 (2016) 18–35.
[21] Q.Z. Kong, W.H. Xu, The comparative study of covering rough sets and multi-granulation rough sets, Soft Comput. 23 (2019) 3237–3251.
[22] W.T. Li, W.H. Xu, Multigranulation decision-theoretic rough set in ordered information system, Fundam. Inform. 139 (1) (2015) 67–89.
[23] J.H. Yu, X.Y. Zhang, Z.H. Zhao, W.H. Xu, Uncertainty measures in multigranulation with different grades rough set based on dominance relation, J. Intell. Fuzzy Syst. 31 (2) (2016) 1133–1144.
[24] X.B. Yang, Y.S. Qi, X.N. Song, J.Y. Yang, Test cost sensitive multigranulation rough set: model and minimal cost selection, Inf. Sci. 250 (11) (2013) 184–199.
[25] H.R. Ju, H.X. Li, X.B. Yang, X.Z. Zhou, H. Bing, Cost-sensitive rough set: a multi-granulation approach, Knowl.-Based Syst. 123 (2017) 137–153.
[26] C.X. Hu, S.X. Liu, G.X. Liu, Matrix-based approaches for dynamic updating approximations in multigranulation rough sets, Knowl.-Based Syst. 122 (2017) 51–63.
[27] J.H. Li, R. Yue, C.L. Mei, Y.H. Qian, X.B. Yang, A comparative study of multigranulation rough sets and concept lattices via rule acquisition, Knowl.-Based Syst. 91 (2016) 152–164.
[28] Y.H. Qian, H. Zhang, Y.L. Sang, J.Y. Liang, Multigranulation decision-theoretic rough sets, Int. J. Approx. Reason. 55 (1) (2014) 225–237.
[29] G.P. Lin, Y.H. Qian, J.J. Li, NMGRS: neighborhood-based multigranulation rough sets, Int. J. Approx. Reason. 53 (7) (2012) 1080–1093.
[30] J. Zhang, X.Y. Zhang, W.H. Xu, Y.X. Wu, Local multigranulation decision-theoretic rough set in ordered information systems, Soft Comput. 23 (24) (2019) 13247–13261.
[31] B.Y. Lin, W.H. Xu, Multi-granulation rough set for incomplete interval-valued decision information systems based on multi-threshold tolerance relation, Symmetry 10 (6) (2018) 208–229.
[32] J.H. Yu, B. Zhang, M.H. Chen, W.H. Xu, Double-quantitative decision-theoretic approach to multigranulation approximate space, Int. J. Approx. Reason. 98 (2018) 236–258.
[33] W.H. Xu, Q.R. Wang, X.T. Zhang, Multi-granulation rough sets based on tolerance relations, J. Intell. Fuzzy Syst. 17 (2013) 1241–1252.
[34] W.H. Xu, Q.R. Wang, S.Q. Luo, Multi-granulation fuzzy rough sets, J. Intell. Fuzzy Syst. 26 (2014) 1323–1340.
[35] J.M. Zhan, W.H. Xu, Two types of coverings based multigranulation rough fuzzy sets and applications to decision making, Artif. Intell. Rev. 3 (2018) 1–32.
[36] J. Qian, C.H. Liu, X.D. Yue, Multigranulation sequential three-way decision based on multiple thresholds, Int. J. Approx. Reason. 105 (2019) 396–416.
[37] B.B. Sang, L. Yang, H.M. Chen, W.H. Xu, Y.T. Guo, Z. Yuan, Generalized multi-granulation double-quantitative decision-theoretic rough set of multi-source information system, Int. J. Approx. Reason. 115 (2019) 157–179.
[38] R. Gravina, P. Alinia, H. Ghasemzadeh, G. Fortino, Multi-sensor fusion in body sensor networks: state-of-the-art and research challenges, Inf. Fusion 35 (2017) 68–80.
[39] B.P. Cai, Y.H. Liu, Q. Fan, Y.W. Zhang, Z.K. Liu, Y.U. Shilin, J.I. Renjie, Multi-source information fusion based fault diagnosis of ground-source heat pump using Bayesian network, Appl. Energy 114 (2) (2014) 1–9.
[40] M.S.A. Karim, K.S. Wong, Data fusion in universal domain using dual semantic code ☆, Inf. Sci. 283 (2014) 123–141.
[41] B.B. Sang, Y.T. Guo, D.R. Shi, W.H. Xu, Decision-theoretic rough set model of multi-source decision systems, Int. J. Mach. Learn. Cybern. 9 (2018) 1941–1954.
[42] X.Y. Jia, Z. Deng, F. Min, D. Liu, Three-way decisions based feature fusion for Chinese irony detection, Int. J. Approx. Reason. 113 (2019) 324–335.
[43] W.H. Xu, J.H. Yu, A novel approach to information fusion in multi-source datasets: a granular computing viewpoint, Inf. Sci. 378 (2017) 410–423.
[44] Y.Y. Huang, T.R. Li, C. Luo, H. Fujita, S.J. Horng, Dynamic fusion of multi-source interval-valued data by fuzzy granulation, IEEE Trans. Fuzzy Syst. 26 (6) (2018) 3403–3417.
[45] Y.T. Guo, W.H. Xu, Attribute reduction in multi-source decision systems, in: International Joint Conference on Rough Sets, Vol. 9920, 2016, pp. 558–568.
[46] M.M. Li, X.Y. Zhang, Information fusion in a multi-source incomplete information system based on information entropy, Entropy 19 (11) (2017) 570–586.
[47] W.H. Xu, M.M. Li, X.Z. Wang, Information fusion based on information entropy in fuzzy multi-source incomplete information system, Int. J. Fuzzy Syst. 19 (4) (2017) 1200–1216.
[48] W. Wei, J.Y. Liang, Information fusion in rough set theory: an overview, Inf. Fusion 48 (2019) 107–118.
[49] G.P. Lin, J.Y. Liang, Y.H. Qian, An information fusion approach by combining multigranulation rough sets and evidence theory, Inf. Sci. 314 (2015) 184–199.
[50] X.Y. Che, J.S. Mi, D.G. Chen, Information fusion and numerical characterization of a multi-source information system, Knowl.-Based Syst. 145 (2018) 121–133.
[51] X.H. Zhang, D.W. Pei, J.H. Dai, Fuzzy Mathematics and Rough Set Theory, Tsinghua University Press, Beijing, 2013.
[52] W.H. Xu, Order Information System and Rough Set, Science Press, Beijing, 2013.